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Pseudo-symmetries, Noether's theorem and the adjoint equation

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Abstract. Pseudo-symmetries were introduced by Sarlet and Cantrijn for time-dependent non-conservative systems. They are reconsidered here in the context of general autonomous second-order systems, relying on the new approach to such systems which was presented by Sarlet *et al.* We further introduce the notion of adjoint symmetries of a second-order system, as being associated to invariant 1-forms, and show how they may be related to first integrals or to Lagrangians under appropriate circumstances. Our results enable us to clarify a rather unusual account of Noether's theorem which was recently given by Gordon.

1. Introduction

In a previous paper (Sarlet *et al* 1984), we have re-analysed the various elements entering the definition of a Lagrangian vector field on the tangent bundle of a differentiable manifold. In doing so, we identified certain geometrical objects which are also of interest outside the scope of Lagrangian mechanics. In particular, we have associated to any second-order equation field Γ on TM a subset $\mathfrak{X}^*_\Gamma$ of 1-forms on TM. Such 1-forms played an important role in establishing a number of results which generalise known properties of Lagrangian systems. Perhaps the most appealing result in that respect was a covering of Noether's theorem, which also contained Cantrijn's analogue of Noether's theorem for non-conservative systems (Cantrijn 1982) as a special case.

The class of vector fields which, in Cantrijn's approach, were put into correspondence with first integrals have been incorporated into a wider class of vector fields, called pseudo-symmetries, by Sarlet and Cantrijn (1984). It was further shown there that a pseudo-symmetry of a non-conservative system, which is not of Noether type, but generates point transformations, quite unexpectedly gives rise to a Lagrangian for the same system. It is our first goal in the present paper to reconsider and extend the results on pseudo-symmetries within the context of general second-order equation fields Γ and their associated 1-forms in $\mathfrak{X}^*_\Gamma$.

In particle mechanics, Noether's theorem has always been regarded as providing a relationship between a subclass of symmetry vector fields of the given dynamical vector field and first integrals (see, e.g., Sarlet and Cantrijn 1981, Marmo *et al* 1985, ch 15). A recent account of Noether's theorem by Gordon (1986) may look rather startling from this point of view, because there is no role for symmetry vector fields

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in Gordon's version. Instead, first integrals are generated, under appropriate circumstances, through solutions of the adjoints of the linear variational equations. So, at best, this can be a kind of dual version of Noether's theorem, and we wish to clarify this in geometrical terms. To that end, we first have to give an intrinsic meaning to the notion of an 'adjoint symmetry'. Since a symmetry is a vector field which is invariant under the given dynamics, one may contemplate defining an adjoint symmetry as being an invariant 1-form. This is fine for first-order systems and in fact has been formally introduced this way, for example, by Ten Eikelder (1984). For second-order equation fields Γ, however, we will show that it is more convenient to reserve the term 'adjoint symmetry', not for an invariant 1-form, but for a related element of the set $\mathfrak{X}^*_\Gamma$. Gordon has rightly claimed that his version of Noether's theorem can give rise to a first integral even when the given system is not of Lagrangian type. It will appear now that this is exactly the dual version of the way we have extended Noether's theorem from symmetries (in the Lagrangian case) to pseudo-symmetries (in the non-Lagrangian case). The interconnection with the first part of the paper will be complete when we establish that the creation of a Lagrangian by a pseudo-symmetry of point type is also related to a more general result on a class of adjoint symmetries. A simple example will serve to illustrate all these matters.

As a final contribution of this paper, we shall introduce a geometrical notion of 'self-adjointness' of a second-order equation field Γ and prove that Γ is self-adjoint if and only if it is locally Lagrangian. This result will have certain advantages over the traditional analytical one, in which a differential equation is said to be self-adjoint if its linear variational equations coincide with their adjoints.

2. Preliminaries

Let Γ denote a second-order equation field on TM, which for all local considerations will be written in the form

$$\Gamma = v^i(\partial/\partial q^i) + \Lambda^i(q, v)\partial/\partial v^i. \tag{1}$$

The tangent bundle TM carries a natural integrable almost tangent structure (see, e.g., Crampin 1983a, b), determined by the intrinsic type (1, 1) tensor field

$$S = (\partial/\partial v^i)\otimes \mathrm{d}q^i. \tag{2}$$

The main properties of S are: $S^2 = 0$, the Nijenhuis tensor of S is zero, and with respect to any second-order equation field Γ we have

$$S \circ \mathscr{L}_\Gamma S = -\mathscr{L}_\Gamma S \circ S = S \tag{3}$$

$$(\mathscr{L}_\Gamma S)^2 = \mathscr{I}. \tag{4}$$

Composition of type (1, 1) tensor fields here is understood as composition of the linear maps on 1-forms they determine. Properties like (3) have to be transposed when the tensors are regarded as maps on vector fields.

We now recall the following definitions from Sarlet *et al* (1984):

$$\mathfrak{X}^*_\Gamma = \{\phi \in \mathfrak{X}^*(TM) | \mathscr{L}_\Gamma(S(\phi)) = \phi\} \tag{5}$$

$$\mathfrak{X}_\Gamma = \{X \in \mathfrak{X}(TM) | S(\mathscr{L}_\Gamma X) = 0\}. \tag{6}$$

Locally, elements of these subsets of 1-forms and vector fields have the following appearance:

$$\phi = \alpha_j \, \mathrm{d}v^j + \Gamma(\alpha_j) \, \mathrm{d}q^j \tag{7}$$

$$X = \mu^i(\partial/\partial q^i) + \Gamma(\mu^i)\partial/\partial v^i. \tag{8}$$

A 1-form $\phi \in \mathfrak{X}^*_\Gamma$ is called non-degenerate if $\mathrm{d}S(\phi)$ is a symplectic form. The given vector field Γ is defined to be Lagrangian (locally Lagrangian) if $\mathfrak{X}^*_\Gamma$ contains an element ϕ which is exact (closed). To each vector field Y we can associate a type $(1, 1)$ tensor field R_Y, defined by

$$R_Y = \mathscr{L}_\Gamma S \circ \mathscr{L}_Y S + S \circ \mathscr{L}_{[\Gamma, Y]} S. \tag{9}$$

Such tensor fields have the properties $[R_Y, S] = 0$ and $\mathscr{L}_\Gamma R_Y \circ S = 0$, which is sufficient to guarantee that they preserve the sets $\mathfrak{X}^*_\Gamma$ and $\mathfrak{X}_\Gamma$.

To end this section, we introduce projection operators for 1-forms and vector fields, both denoted by π_Γ, as follows:

$$\pi_\Gamma : \mathfrak{X}^*(TM) \to \mathfrak{X}^*_\Gamma \qquad \alpha \mapsto \pi_\Gamma(\alpha) = \mathscr{L}_\Gamma(S(\alpha)) \tag{10}$$

$$\pi_\Gamma : \mathfrak{X}(TM) \to \mathfrak{X}_\Gamma \qquad Y \mapsto \pi_\Gamma(Y) = Y + S(\mathscr{L}_\Gamma Y). \tag{11}$$

It is straightforward to verify that these are indeed projections and that, for example, the following property holds true:

$$R_Y = R_{\pi_\Gamma(Y)} = \mathscr{L}_\Gamma S \circ \mathscr{L}_{\pi_\Gamma(Y)} S. \tag{12}$$

3. Pseudo-symmetries

Sarlet and Cantrijn (1984) introduced pseudo-symmetries, in the context of time-dependent mechanics, for systems governed by a 'conservative part' with Lagrangian L and additional non-conservative forces Q_i. Within the present framework of autonomous second-order equations, such systems are characterised by the fact that $\mathfrak{X}^*_\Gamma$ contains an element of the form

$$\phi = \mathrm{d}L + Q_i(q, v) \, \mathrm{d}q^i. \tag{13}$$

Because there is a marked difference between formulae ruling time-dependent dynamical systems on the odd-dimensional manifold $\mathscr{R} \times TM$ and their counterparts for autonomous systems on the even-dimensional manifold TM, we wish to reconsider the concept of pseudo-symmetries here. Moreover we intend to define them for a general second-order equation field Γ, with respect to a corresponding element of $\mathfrak{X}^*_\Gamma$ which need not be of the specific form (13).

Let ϕ denote a non-degenerate element of $\mathfrak{X}^*_\Gamma$.

Definition. $Y \in \mathfrak{X}(TM)$ is a pseudo-symmetry of Γ with respect to ϕ if

$$i_{[Y,\Gamma]} \, \mathrm{d}S(\phi) = i_Y \, \mathrm{d}\phi. \tag{14}$$

It is clear from this definition that, in the case of a Lagrangian vector field ($\phi = \mathrm{d}L$), we are actually talking about symmetries of Γ. The left-hand side of (14) can be rewritten as follows:

$$i_Y \mathscr{L}_\Gamma \, \mathrm{d}S(\phi) - \mathscr{L}_\Gamma i_Y \, \mathrm{d}S(\phi) = i_Y \, \mathrm{d}\phi - \mathscr{L}_\Gamma i_Y \, \mathrm{d}S(\phi)$$

from which we see that Y is a pseudo-symmetry with respect to ϕ if and only if

$$\mathscr{L}_\Gamma i_Y \, \mathrm{d}S(\phi) = 0. \tag{15}$$

Definition. $Y \in \mathfrak{X}(TM)$ is a pseudo-symmetry of Noether type (with respect to ϕ) if $\mathscr{L}_Y(S(\phi)) = \mathrm{d}f$, for some function f, and $i_Y(\phi - \mathrm{d}\langle\Delta, \phi\rangle) = 0$.

Here $\Delta = S(\Gamma) = v^i(\partial/\partial v^i)$ is the so-called dilation vector field. It is straightforward to verify that such a Y is indeed a pseudo-symmetry, i.e. satisfies (15). In addition, we have $i_Y\,\mathrm{d}S(\phi) = \mathrm{d}F$, where $F = f - \langle Y, S(\phi)\rangle$ and $\Gamma(F) = 0$. In this way, we recover the generalisation of Noether's theorem discussed in Sarlet *et al* (1984).

Now let Y be a general pseudo-symmetry with respect to $\phi \in \mathfrak{X}^*_\Gamma$. We know that Y determines a type (1, 1) tensor field R_Y and we may expect to gain some further information by looking at the new element of $\mathfrak{X}^*_\Gamma$ which $R_Y(\phi)$ provides. We have

$$\begin{aligned} R_Y(\phi) &= (\mathscr{L}_\Gamma S \circ \mathscr{L}_Y S)(\phi) + S(\mathscr{L}_{[\Gamma, Y]}(S(\phi))) \\ &= (\mathscr{L}_\Gamma S \circ \mathscr{L}_Y S)(\phi) + S(i_{[\Gamma, Y]}\,\mathrm{d}S(\phi) + \mathrm{d}i_{[\Gamma, Y]}S(\phi)). \end{aligned}$$

Using (14), this can be rewritten in the form

$$R_Y(\phi) = (\mathscr{L}_\Gamma S \circ \mathscr{L}_Y S)(\phi) - S(\mathscr{L}_Y\phi) + S(\mathrm{d}L')$$

where the function L' is defined by

$$L' = \langle Y + S([\Gamma, Y]), \phi\rangle = \langle \pi_\Gamma(Y), \phi\rangle. \tag{16}$$

Acting on both sides by $\mathscr{L}_\Gamma S$ and using the properties (3) and (4), it follows that

$$(\mathscr{L}_\Gamma S \circ R_Y)(\phi) = \mathscr{L}_Y(S(\phi)) - S(\mathrm{d}L'). \tag{17}$$

A case of special interest is the case where the pseudo-symmetry Y is of point type, i.e. projects onto a vector field on M. In such a case, indeed, we have $R_Y = 0$, and by taking the Lie derivative of (17) with respect to Γ we obtain

$$\begin{aligned} \pi_\Gamma(\mathrm{d}L') &= \mathscr{L}_\Gamma \mathscr{L}_Y(S(\phi)) \\ &= \mathscr{L}_{[\Gamma, Y]}(S(\phi)) + \mathscr{L}_Y \mathscr{L}_\Gamma(S(\phi)) \\ &= \mathrm{d}i_{[\Gamma, Y]}S(\phi) + i_{[\Gamma, Y]}\,\mathrm{d}S(\phi) + \mathscr{L}_Y\phi \\ &= \mathrm{d}\langle S([\Gamma, Y]), \phi\rangle - i_Y\,\mathrm{d}\phi + \mathscr{L}_Y\phi \\ &= \mathrm{d}L' \end{aligned}$$

which proves that $\mathrm{d}L' \in \mathfrak{X}^*_\Gamma$. We thus reach the following conclusion.

Proposition 1. If Y is a pseudo-symmetry of point type (with respect to ϕ), then $L' = \langle \pi_\Gamma(Y), \phi\rangle$ is a Lagrangian for the given second-order system Γ (L' need not necessarily be regular, however).

This result covers the familiar property that a point symmetry of a Lagrangian system produces an alternative Lagrangian. The appearance of a Lagrangian L' in the present context of course is rather surprising. Note that a different expression for L', obtained from (16) through some elementary manipulations, is

$$L' = \mathscr{L}_\Gamma i_Y S(\phi). \tag{18}$$

If the pseudo-symmetry Y of proposition 1 in addition happens to be of Noether type, we have $\mathscr{L}_Y(S(\phi)) = \mathrm{d}f$ for some function f, from which it follows that $S(\mathrm{d}f) = (S \circ \mathscr{L}_Y S)(\phi)$. It is easy to see, for example in coordinates, that $S \circ \mathscr{L}_Y S = 0$ when Y projects onto a vector field on M. There results that $S(\mathrm{d}f) = 0$, which means that f is a function on the base manifold M. Moreover, we then know that $\Gamma(F)=0$ with $F=f-i_Y S(\phi)$. Consequently, $L'=\mathscr{L}_\Gamma i_Y S(\phi)=\Gamma(F)=\dot{f}$. So, in agreement with, and as a generalisation of results from, the Lagrangian theory, proposition 1 yields a trivial result in the Noether case (the Lagrangian L' just being a total time derivative of a function of the q^i alone).

Let us now return to the general relation (17), which is valid for any pseudo-symmetry Y. The left-hand side can be rewritten in the following form:

$$(\mathscr{L}_\Gamma S\circ R_Y)(\phi)=\pi_\Gamma(R_Y(\phi))-S(\mathscr{L}_\Gamma(R_Y(\phi)))$$
$$=R_Y(\phi)-S(\mathscr{L}_\Gamma(R_Y(\phi)))$$

since $R_Y(\phi)$ belongs to $\mathfrak{X}^*_\Gamma$. As a result, if we repeat the calculation performed for the case $R_Y=0$, i.e. if we take the Lie derivative of (17) with respect to Γ, we will reach the following more general conclusion.

Proposition 2. If Y is a pseudo-symmetry with respect to ϕ, then the 1-form $\mathrm{d}L'-\mathscr{L}_\Gamma(R_Y(\phi))$ belongs to $\mathfrak{X}^*_\Gamma$, where $L'=\mathscr{L}_\Gamma i_Y S(\phi)$.

Pseudo-symmetries which are not of point type will generally not lead to a Lagrangian. Yet, it appears from the above result that, under certain circumstances, a Lagrangian L' can be obtained. Indeed, we know that $R_Y(\phi)\in\mathfrak{X}^*_\Gamma$ and if it happens also that $\mathscr{L}_\Gamma(R_Y(\phi))\in\mathfrak{X}^*_\Gamma$ (the meaning of which will become clear in the next section), proposition 2 tells us that $\mathrm{d}L'\in\mathfrak{X}^*_\Gamma$. It is this broader possibility which means that we can in fact make a converse statement. The next result generalises the property that two equivalent Lagrangians define a dynamical symmetry of the given system, the local determination of which is a matter of finding a particular solution of a single partial differential equation (Sarlet 1983).

Proposition 3. Let $\phi\in\mathfrak{X}^*_\Gamma$ be non-degenerate and assume further that we have at our disposal a Lagrangian L' for the second-order equation field Γ (i.e. $\mathrm{d}L'\in\mathfrak{X}^*_\Gamma$). Then there exists a related pseudo-symmetry Y with respect to ϕ, which is locally determined by a particular solution F of the equation $\Gamma(F)=L'$. Y is unique to within an arbitrary pseudo-symmetry of Noether type.

Proof. Consider the 1-form $\mathrm{d}F-\theta_{L'}$, where $\theta_{L'}=S(\mathrm{d}L')$ is the Cartan 1-form associated to L'. If F solves the equation $\Gamma(F)=L'$ (possibly only locally), then $\mathscr{L}_\Gamma(\mathrm{d}F-\theta_{L'})=0$. Consequently the relation

$$i_Y\,\mathrm{d}S(\phi)=\mathrm{d}F-\theta_{L'} \tag{19}$$

which, by the fact that $\mathrm{d}S(\phi)$ is symplectic, uniquely defines a vector field Y, yields a pseudo-symmetry in view of (15). If F' were a second solution of $\Gamma(F)=L'$, the difference $F-F'$ would be a first integral of Γ and as such give rise to a pseudo-symmetry of Noether type.

A number of results from this section will make their appearance again in the next section, regarded, however, from quite a different angle.

4. Adjoint symmetries

Symmetries of vector fields are, roughly speaking, invariant vector fields. In defining the notion of an adjoint symmetry it is natural to expect that an invariant 1-form will be involved. As mentioned in the introduction, however, when dealing with second-order equations, there are good reasons for reserving the label 'adjoint symmetry' not for an invariant 1-form as such, but for a related 1-form. We will first present our definition and an alternative characterisation of adjoint symmetries and comment on the motivation for doing so afterwards.

Definition. An adjoint symmetry of a second-order equation field Γ is a 1-form α on TM with the property that $\beta = \mathscr{L}_\Gamma S(\alpha)$ is invariant, i.e. $\mathscr{L}_\Gamma \beta = 0$.

Proposition 4. A 1-form α is an adjoint symmetry of Γ if and only if $\alpha \in \mathfrak{X}^*_\Gamma$ and $\mathscr{L}_\Gamma \alpha \in \mathfrak{X}^*_\Gamma$.

Proof. If α is an adjoint symmetry, we have (using (4))

$$\alpha = \mathscr{L}_\Gamma S(\beta) = \pi_\Gamma(\beta)$$

since $\mathscr{L}_\Gamma \beta = 0$. This means that $\pi_\Gamma(\alpha) = \alpha$ or $\alpha \in \mathfrak{X}^*_\Gamma$. Furthermore, we can write

$$\beta = \mathscr{L}_\Gamma S(\alpha) = \pi_\Gamma(\alpha) - S(\mathscr{L}_\Gamma \alpha)$$

and this implies

$$0 = \mathscr{L}_\Gamma \beta = \mathscr{L}_\Gamma \alpha - \pi_\Gamma(\mathscr{L}_\Gamma \alpha)$$

i.e. $\mathscr{L}_\Gamma \alpha \in \mathfrak{X}^*_\Gamma$. The converse is obvious from the same calculations.

Locally, with a Γ of the form (1), the condition that α be an adjoint symmetry is that it first of all be of the form (7), as an element of $\mathfrak{X}^*_\Gamma$, and that the coefficients α_j in (7) satisfy the equations

$$\Gamma\Gamma(\alpha_j) + \Gamma\left(\alpha_k \frac{\partial \Lambda^k}{\partial v^j}\right) - \alpha_k \frac{\partial \Lambda^k}{\partial q^j} = 0. \tag{20}$$

These are second-order partial differential equations for the n functions α_j, but they bear a close resemblance to the ordinary second-order differential equations which are known as the adjoint equations of the linear variational equations of $\ddot{q}^i = \Lambda^i(q, \dot{q})$. For a detailed discussion of such matters we can refer, for example, to Cantrijn *et al* (1987). Here, we content ourselves with observing that the appearance of equations (20) is sufficient for a justification of the term adjoint symmetry, as introduced above.

We now come to a characterisation of an interesting subclass of adjoint symmetries, which will help in understanding from a different angle all results on pseudo-symmetries of the previous section (and of course many known results for Lagrangian systems which they cover). The subclass in question concerns those adjoint symmetries which are projections under π_Γ of an exact 1-form.

Proposition 5. Let α be an adjoint symmetry of Γ which is of the form $\alpha = \pi_\Gamma(\mathrm{d}F)$ for some function F: then $\mathrm{d}\Gamma(F) \in \mathfrak{X}^*_\Gamma$ (i.e. either F provides a first integral or $\Gamma(F)$ is a possibly degenerate Lagrangian for Γ). Conversely, if $\mathrm{d}\Gamma(F) \in \mathfrak{X}^*_\Gamma$ for some function F, then $\alpha = \pi_\Gamma(\mathrm{d}F)$ is an adjoint symmetry of Γ.

Proof. We have

$$\alpha = \pi_\Gamma(\mathrm{d}F) = \mathscr{L}_\Gamma S(\mathrm{d}F) + S(\mathrm{d}\Gamma(F)).$$

Using properties (3) and (4), this implies

$$\mathscr{L}_\Gamma S(\alpha) = \mathrm{d}F - S(\mathrm{d}\Gamma(F)).$$

This relation makes it clear that $\mathscr{L}_\Gamma(\mathscr{L}_\Gamma S(\alpha)) = 0$ if and only if $\pi_\Gamma(\mathrm{d}\Gamma(F)) = \mathrm{d}\Gamma(F)$, from which the results now readily follow.

The theory on pseudo-symmetries of the previous section in fact provides a nice application of proposition 5. To see this, we first make some more general comments. We know, from the very definition, that every element of $\mathfrak{X}^*_\Gamma$ is the Lie derivative with respect to Γ of a semi-basic 1-form. It is, however, not excluded that such an element might further be written in a form $\mathscr{L}_\Gamma \sigma$ with a σ which is not semi-basic. If now $\mathscr{L}_\Gamma \sigma \in \mathfrak{X}^*_\Gamma$, we have a 1-form σ which satisfies half of the requirements (see proposition 4) for an adjoint symmetry and we may wonder whether we can construct an adjoint symmetry σ' out of σ by, for example, projecting σ onto $\mathfrak{X}^*_\Gamma$. Putting $\sigma' = \pi_\Gamma(\sigma) \in \mathfrak{X}^*_\Gamma$, the difference $\sigma - \sigma'$ is semi-basic (it is obtained by acting on σ with the Euler-Lagrange operator) and so $\mathscr{L}_\Gamma(\sigma - \sigma') \in \mathfrak{X}^*_\Gamma$. Since by assumption $\mathscr{L}_\Gamma \sigma \in \mathfrak{X}^*_\Gamma$ also, the same property holds for σ', meaning that σ' is an adjoint symmetry. For this construction to make sense, however, it is essential that the original 1-form σ is not semi-basic, for otherwise $\sigma' = \pi_\Gamma(\sigma) \equiv 0$.

To come back to the previous section, assume we have a 1-form $\phi \in \mathfrak{X}^*_\Gamma$ (non-degenerate) and let Y for the time being be any vector field on TM. One can show then that $\rho = \mathscr{L}_Y(S(\phi)) - R_Y(\phi)$ is semi-basic. Therefore, $\mathscr{L}_\Gamma \rho \in \mathfrak{X}^*_\Gamma$. The special interest of Y being a pseudo-symmetry (with respect to ϕ) is that $i_Y\, \mathrm{d}S(\phi)$ is an invariant 1-form. So, by subtracting it from ρ, we still have a 1-form σ whose Lie derivative with respect to Γ belongs to $\mathfrak{X}^*_\Gamma$. This time, however, σ is no longer semi-basic so that the above construction applies and tells us that

$$\sigma' = \pi_\Gamma(\mathrm{d}i_Y S(\phi) - R_Y(\phi)) = \pi_\Gamma(\mathrm{d}i_Y S(\phi)) - R_Y(\phi) \tag{21}$$

is an adjoint symmetry of Γ. Knowing proposition 5, we can now easily recover the results of propositions 1 and 2. Indeed, if the pseudo-symmetry Y is of point type, i.e. $R_Y = 0$ (see proposition 1), or more generally if $R_Y(\phi)$ itself happens to be an adjoint symmetry (see the discussion following proposition 2), then it follows from (21) and proposition 5 that $L' = \Gamma(i_Y S(\phi))$ is a Lagrangian for the second-order equation field Γ. The other particular case of interest, namely the case of pseudo-symmetries of Noether type, also fits well into the framework of proposition 5. One can easily verify, indeed, that for a pseudo-symmetry of Noether type, the associated adjoint symmetry σ' of (21) can be written in the form $\sigma' = -\pi_\Gamma(\mathrm{d}F)$, where $F = f - \langle Y, S(\phi) \rangle$ is the corresponding first integral discussed in the previous section.

5. Gordon's version of Noether's theorem

Noether's theorem, in practically all versions one can find in the literature (including the original one by Noether (1918)), deals with a relationship between symmetry vector fields of a Lagrangian system and first integrals. Gordon's recent version of Noether's theorem (Gordon 1986) at first glance may create the impression of having nothing to do with Noether's theorem whatsoever. The reason is that Gordon essentially does not mention symmetry vector fields at all. Instead, he relates first integrals to certain solutions of the 'adjoint equations', which for the type of second-order systems (1) we are presently discussing, are precisely the partial differential equations (20). The theory developed in the previous sections allows us to clarify the contents of Gordon's statement from a geometrical point of view and to add some interesting features to it. For the sake of comparison, we will refer here to the notation of Gordon (1986) and point out what they stand for in our terminology. For completeness, we should mention that the statement we are quoting here is actually a particular case of a more general formulation which is valid for systems of partial differential equations also (Gordon 1984). According to Gordon, Noether's theorem takes the following appearance:

$$D_tQ \equiv \lambda f \Rightarrow D_t\tilde{Q}[\sigma] = \lambda\tilde{f}[\sigma] \Leftrightarrow \tilde{f}^*[\lambda] = 0 \tag{22}$$

where the first arrow can only be extended to an arrow in both directions if a certain integrability condition is satisfied. The left-hand side of (22) becomes $\Gamma(Q) = 0$, i.e. it merely expresses that Q is a first integral. The reason why Gordon does not write $D_tQ = 0$ is simply that his equation of motion is represented by $f(x, \dot{x}, \ddot{x}, t) = 0$, while we write the given second-order equations in normal form. Note in passing that Gordon discusses a single second-order equation (possibly time dependent), while we deal with a system of n autonomous second-order equations, but these differences appear to be irrelevant. It suffices to think of the f in (22) as representing a vector with components $f^i = \ddot{q}^i - \Lambda^i(q, \dot{q})$ (with some additional adaptations which will be mentioned if need be). The right-hand side of (22) represents the adjoint equations. In other words, with λ being thought of as a vector with components α_j, the relation $\tilde{f}^*[\lambda] = 0$ must be interpreted as standing exactly for equation (20). Thus, a solution of that equation for us represents a certain 1-form $\alpha = \alpha_j\, dv^j + \Gamma(\alpha_j)\, dq^j \in \mathscr{X}^*_\Gamma$.

The middle part of (22) is difficult to translate in direct terms. In Gordon's terminology it stands for an 'infinitesimal invariance'. When $\tilde{Q}[\sigma]$ is generated out of the function Q, it takes the form (see Gordon's equation (2.5), here slightly adapted to our notation)

$$\tilde{Q}[\sigma] = (\partial Q/\partial q^i)\sigma^i + (\partial Q/\partial v^i)\Gamma(\sigma^i). \tag{23}$$

This sufficiently illustrates that $\tilde{Q}$, in geometrical terms, must be thought of as a 1-form. In (23) it is actually the 1-form dQ, which is paired with the vector field $\sigma^i(\partial/\partial q^i) + \Gamma(\sigma^i)\partial/\partial v^i \in \mathscr{X}_\Gamma$. When, on the other hand, $\tilde{Q}[\sigma]$ is constructed out of a solution of the adjoint equations, it takes the form (see Gordon's equation (2.6) with appropriate adaptations)

$$\tilde{Q}[\sigma] = \left(-\frac{\partial\Lambda^j}{\partial v^i}\alpha_j - \Gamma(\alpha_i)\right)\sigma^i + \alpha_i\Gamma(\sigma^i). \tag{24}$$

In that case, therefore, $\tilde{Q}$ is the 1-form

$$\tilde{Q} = \left(-\frac{\partial\Lambda^j}{\partial v^i}\alpha_j - \Gamma(\alpha_i)\right) dq^i + \alpha_i\, dv^i$$

which is exactly $\mathscr{L}_\Gamma S(\alpha)$. The middle part of (22), therefore, cannot represent anything but the invariance of that 1-form and so the meaning of the double arrow in (22) is obvious in terms of our notion of adjoint symmetry. The integrability condition which Gordon imposes (his equation (2.7)) is just the identification of (24) with (23). In other words, it represents the requirement that $\mathscr{L}_\Gamma S(\alpha)$ be an exact form. The content of (22) now is perfectly clear and is incorporated in our results of the previous section. Indeed, (22) says:

(i) if F is a first integral of Γ, then $\mathrm{d}F$ is an invariant 1-form, which can be written in the form $\mathscr{L}_\Gamma S(\alpha)$, where $\alpha = \pi_\Gamma(\mathrm{d}F)$ is an adjoint symmetry;

(ii) conversely, if α is an adjoint symmetry, then $\mathscr{L}_\Gamma S(\alpha)$ is an invariant 1-form. If, in addition, this 1-form satisfies the 'integrability condition' $\mathscr{L}_\Gamma S(\alpha) = \mathrm{d}F$, we have $\mathrm{d}\Gamma(F) = 0$ and therefore $\Gamma(F) = c$ (a constant). That we do not immediately find a first integral in this case is a consequence of our restriction to an autonomous theory (obviously, $F - ct$ is a first integral).

With regard to (ii) it is interesting to note that we have in fact identified (in proposition 5) some kind of weaker 'integrability condition' that can be imposed on an adjoint symmetry, namely, instead of imposing $\mathscr{L}_\Gamma S(\alpha) = \mathrm{d}F$ in which case $\alpha = \pi_\Gamma(\mathrm{d}F)$, we could impose only the latter. The function F will then not necessarily give rise to a first integral, but $\Gamma(F)$ will be a Lagrangian. All such aspects will be illustrated for a simple example in the next section.

In conclusion, Gordon's statement should really not be called Noether's theorem; it concerns interesting properties of adjoint symmetries of general second-order equations. However, it does become a perfectly valid dual description of Noether's theorem whenever there is a Lagrangian L available, the dualism being provided by the isomorphism between vector fields and 1-forms defined by the symplectic form $\mathrm{d}\theta_L$. Even more generally, it becomes a dual description of our theory on pseudo-symmetries whenever we know a non-degenerate $\phi \in \mathfrak{X}^*_\Gamma$, the dualism this time being generated by the symplectic form $\mathrm{d}S(\phi)$.

The above discussion is suggestive for introducing a geometrical notion of 'self-adjointness' of a second-order equation field Γ. In the analytical approach a second-order system is said to be self-adjoint if its linear variational equations coincide with their adjoints (see, e.g., Santilli 1978). It is then shown that self-adjointness is necessary and sufficient for the system to be Lagrangian. In our present context, symmetries and adjoint symmetries are geometrical quantities of a different species and cannot possibly be said to 'coincide'. Inspired by what preceeds, however, we are led to introducing the following concept.

Definition. A second-order equation field Γ is said to be self-adjoint if there exists a non-degenerate $\phi \in \mathfrak{X}^*_\Gamma$, such that the isomorphism $\gamma : \mathfrak{X}(TM) \to \mathfrak{X}^*(TM)$, defined by

$$\gamma : Y \mapsto \mathscr{L}_\Gamma S(i_Y \, \mathrm{d}S(\phi)) \tag{25}$$

provides a bijection between symmetries of Γ and adjoint symmetries.

As an intermediate remark, elements of $\mathfrak{X}^*_\Gamma$ need not exist globally and Γ need not have any global symmetries Y. The content of the above definition and its interpretation below is therefore in the first place a local one and the same is true for many other results in this paper.

Proposition 6. Γ is self-adjoint if and only if Γ is locally Lagrangian.

Proof.

(i) If we set $\alpha = \gamma(Y)$ and $\beta = \mathscr{L}_\Gamma S(\alpha)$, we have $i_Y \, \mathrm{d}S(\phi) = \beta$, from which it follows that

$$\begin{aligned}\mathscr{L}_\Gamma \beta &= i_Y \mathscr{L}_\Gamma \, \mathrm{d}S(\phi) + i_{[\Gamma, Y]} \, \mathrm{d}S(\phi) \\ &= i_Y \, \mathrm{d}\phi + i_{[\Gamma, Y]} \, \mathrm{d}S(\phi).\end{aligned}$$

Now, α is an adjoint symmetry if and only if $\mathscr{L}_\Gamma\beta = 0$, while Y is a symmetry if and only if $i_{[\Gamma,Y]}\,\mathrm{d}S(\phi) = 0$. We thus reach the conclusion that Γ is self-adjoint if and only if there exists a non-degenerate $\phi \in \mathfrak{X}^*_\Gamma$ such that $i_Y\,\mathrm{d}\phi = 0$ for all symmetries Y of Γ.

(ii) We next prove: if for a 2-form ω on TM we have that $i_Y\omega = 0$ for all symmetries Y of a given second-order equation field Γ, then $\omega = 0$. Let (q, v) be a regular point of Γ. Then, in the neighbourhood of this point there exists a coordinate transformation $(q^i, v^i) \mapsto (x^k)_{1 \leq k \leq 2n}$, such that Γ in these coordinates simply becomes $\partial/\partial x^1$. But then all basis vector fields $\partial/\partial x^k$ are symmetries of Γ in these coordinates and therefore ω must be zero in the neighbourhood under consideration. Singular points of Γ all lie in the zero section of TM and therefore can be approached arbitrarily close by regular points. It follows by continuity that $\omega = 0$ everywhere.

From (i) and (ii) we conclude that Γ is self-adjoint if and only if there exists a non-degenerate $\phi \in \mathfrak{X}^*_\Gamma$, such that $\mathrm{d}\phi = 0$, which means that Γ is locally Lagrangian.

The above formulation of self-adjointness appears to have some advantages over the classical analytical one. Indeed, classically, when one states that the variational equations and their adjoints coincide, it follows that the given system is a set of Euler–Lagrange equations as it stands. We may be talking, for example, of a system written in the form $a_{ij}(\ddot{q}^j - \Lambda^j) = 0$ and self-adjointness then is a property related to the pre-assigned matrix (a_{ij}). In other words, the formalism does not account for the possibility that a system which fails to be self-adjoint in the given form may still be derivable from a Lagrangian if one passes to an equivalent description, i.e. one with a different (regular) multiplier matrix (a_{ij}). Our definition of self-adjointness relates to the normal form of the equations and therefore does not start from a preassigned matrix (a_{ij}). We have shown that if Γ happens to be self-adjoint then it can be derived, at least locally, from a Lagrangian L, whose Hessian will constitute the matrix (a_{ij}) for which the standard definition of self-adjointness works. Our formalism covers in itself the possibility that there may be more than one ϕ fitting the description, leading therefore to different Lagrangians for the same system.

6. An example

We take a simple example of a single second-order equation, which was also considered by Gordon and thus will allow us to illustrate various points made in § 5.

Consider the equation

$$\ddot{q} = -q\dot{q}^2 \tag{26}$$

which corresponds to the second-order equation field $\Gamma = v(\partial/\partial q) - qv^2(\partial/\partial v)$. Its adjoint linear equation (20) becomes

$$\Gamma\Gamma(\lambda) - 2qv\Gamma(\lambda) + (2q^2 - 1)v^2\lambda = 0. \tag{27}$$

Gordon found the particular solution $\lambda = v^{-1}$ of (27), leading to the first integral $G = \frac{1}{2}q^2 + \ln v$. As discussed in the previous section, we do not regard the process involved in obtaining this G as Noether's theorem. Since at the moment we do not have a Lagrangian for (26) at our disposal, closer to Noether's theorem would in fact be a relationship between G and a pseudo-symmetry Y with respect to some $\phi \in \mathfrak{X}^*_\Gamma$ (see § 3). To illustrate this point, let us regard the right-hand side of (26) as some non-conservative force $Q=-qv^2$, interacting with the free particle Lagrangian $L=\frac{1}{2}v^2$. In other words, we are focusing on the following 1-form in $\mathfrak{X}^*_\Gamma$:

$$\phi = \mathrm{d}L + Q\,\mathrm{d}q = v\,\mathrm{d}v - qv^2\,\mathrm{d}q. \tag{28}$$

Then $\mathrm{d}S(\phi)=\mathrm{d}v\wedge\mathrm{d}q$ is clearly a symplectic form and the relation

$$i_Y\,\mathrm{d}S(\phi)=\mathrm{d}G$$

associates to the first integral G the pseudo-symmetry of Noether type:

$$Y=-v^{-1}(\partial/\partial q)+q(\partial/\partial v).$$

Let us next seek another particular solution of the adjoint equation by requiring that λ be a function of q only. Equation (27) then reduces to

$$\lambda''-3q\lambda'+(2q^2-1)\lambda=0$$

which can be rewritten in the form

$$(\mathrm{d}/\mathrm{d}q)(\lambda'-q\lambda)-2q(\lambda'-q\lambda)=0$$

and easily integrates to

$$\lambda(q)=c_2u(q)+c_1u(q)\int^q u(q')\,\mathrm{d}q'$$

where $u(q)=\exp(\frac{1}{2}q^2)$ and c_1, c_2 are arbitrary constants. The adjoint symmetry determined by (29) is the 1-form

$$\alpha=\lambda(q)\,\mathrm{d}v+\lambda'(q)v\,\mathrm{d}q$$

and the associated invariant 1-form $\mathscr{L}_\Gamma S(\alpha)$ is

$$\mathscr{L}_\Gamma S(\alpha)=\lambda(q)\,\mathrm{d}v+(2qv\lambda-\lambda'v)\,\mathrm{d}q.$$

Requiring this to be exact, $\mathscr{L}_\Gamma S(\alpha)=\mathrm{d}F$ say (Gordon's integrability condition), leads to the restriction $\lambda'-q\lambda=0$ or $c_1=0$. We are left with the solution $\lambda=u(q)$ (choose $c_2=1$) and the function F is found to be $F=u(q)v$, which is another first integral (not independent of the first one, since $G=\ln F$).

At this point, it is of interest to impose on the adjoint symmetry α the weaker 'integrability condition' $\alpha=\pi_\Gamma(\mathrm{d}F)$ for some function F. In this one degree of freedom case, this is in fact no restriction at all. It merely requires that $\lambda(q)=\partial F/\partial v$, which is trivially satisfied by $F=\lambda(q)v$. Computing $\Gamma(F)$ we find

$$\Gamma(F)=c_1v^2\exp(q^2)$$

which, according to proposition 5, must be a Lagrangian for (26) and indeed is. Let us finally illustrate proposition 3. Choosing for convenience $c_1=\frac{1}{2}$, we have the Lagrangian $L'=\frac{1}{2}v^2\exp(q^2)$. The invariant 1-form is $\mathrm{d}F-\theta_{L'}$ and the relation (19) uniquely determines a pseudo-symmetry Y with respect to ϕ, which is found to be

$$Y=-\lambda\frac{\partial}{\partial q}+\left(\lambda'v-\frac{\partial L'}{\partial v}\right)\frac{\partial}{\partial v}$$

and is evidently a pseudo-symmetry of point type.

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